

INVARIANT HILBERT SCHEMES AND LUNA'S ETALE SLICE THEOREM

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ABSTRACT. Luna's etale slice theorem is a useful theorem for the local study of quotients by reductive algebraic groups. In this article, we show that the slice theorem can also be used to study local structures of invariant Hilbert schemes. By using this method, we show some results on smoothness of invariant Hilbert schemes at closed orbits.

1. INTRODUCTION

Let G be a reductive algebraic group over an algebraically closed field k of characteristic zero. The *Invariant Hilbert scheme* is the scheme parameterizing G -stable closed subschemes of an affine G -scheme X of finite type k . Here a G -scheme means a scheme with an action of G . It is introduced by Alexeev and Brion in [AB05] for the case G is connected. The existence of invariant Hilbert schemes for non connected G is proved in [Br11].

In [MS10] Maclagan and Smith proved that multigraded Hilbert schemes for polynomial ring $\mathbb{Z}[x, y]$ are smooth and irreducible. Therefore invariant Hilbert schemes are smooth and irreducible when G is diagonalizable and X is the affine two space \mathbb{A}_k^2 (in this case any action is isomorphic to a linear action).

Even connectedness is broken for invariant Hilbert schemes which holds for classical Hilbert schemes of projective spaces. It is expected, however, that in low dimensional cases or under good conditions of the action which enable us to reduce problems to lower dimensional cases, some geometric properties can be shown in general.

In this article, we show that Luna's etale slice theorem is compatible with invariant Hilbert schemes. If $x \in X(k)$ is a smooth point such that the orbit Gx is closed, we show that there are following two etale morphisms by using the slice theorem.

$$\begin{array}{ccc}
 & & \text{Hilb}_h^G(X) \\
 & \nearrow & \\
 \text{Hilb}_{h'}^{G_x}(V) & & \\
 & \searrow & \\
 & & \text{Hilb}_h^{G_x}(T_x V)
 \end{array}$$

Here V is a slice at x , G_x is the stabilizer of x , h is a given Hilbert function with $h(0) = 1$ (0 denotes the trivial representation) and h' is a suitable Hilbert function.

The purpose of this paper is to study these morphisms in detail (see Theorem 2.4, Remark 3.6).

If Z is a G -stable closed subscheme of X with underlying topological space Gx and h is the Hilbert function of Z , then h' has finite support. In this case $\text{Hilb}_{h'}^{G_x}(T_x V)$ is an open subscheme of $\text{Hilb}_n(T_x V)^{G_x}$ where $n = \sum_{M \in \text{Irr}(G_x)} h'(M)$ and $\text{Hilb}_n(T_x V)$ is the punctual Hilbert scheme. Therefore, for example, $\text{Hilb}_h^G(X)$ is smooth at Z if $\text{Hilb}_n(T_x V)$ is smooth.

In section 2 we state the main theorems and in section 3 we give the proofs.

2. THE MAIN THEOREMS

Throughout this article, k is an algebraically closed field of characteristic zero, G is a reductive algebraic group over k . Let $\text{Irr}(G)$ denote the set of isomorphism classes of irreducible representations of G , and the class of the trivial representation is denoted by 0. A variety over k means a reduced separated scheme of finite type over k . A G -variety (resp. G -scheme) means a variety (resp. scheme) over k with an action of G .

We use the following well known theorems by Luna (see [Lu73] or [Dr]).

Proposition 2.1. (*Luna's etale slice theorem*)

Let X be an affine G -variety. Let $x \in X(k)$ be a point such that the orbit Gx is closed. Then G_x is reductive and there exist a locally closed subvariety V of X such that

- (1) V is affine and contains x .
- (2) V is G_x -stable.
- (3) *The G -morphism $\psi : G \times^{G_x} V \rightarrow X$ induced by the action of G on X is strongly etale and the image U is a saturated open affine (see Definition 3.1 for the definition of strongly etale morphisms and saturated open sets).*

We call such V an etale slice at x for the action G on X .

There is a more precise version of this theorem when X is smooth at x .

Proposition 2.2. *Under the situation of Proposition 2.1, if X is smooth at x , V can be chosen smooth and there exists a strongly etale G_x morphism $\varphi : V \rightarrow T_x V$ such that*

- (4) $\varphi(x) = 0$, and $(d\varphi)_x = \text{id}$.
- (5) $T_x X = T_x(Gx) \oplus T_x V$.

Luna's etale slice theorem can be used to analyze the local structure of quotients. Since invariant Hilbert schemes can be regarded as "refined quotients", we expect that the slice theorem is effective to study the local structures of invariant Hilbert schemes.

Theorem 2.3. *Let $h : \text{Irr}(G) \rightarrow \mathbb{N}$ be a Hilbert function with $h(0) = 1$. In the situation of Proposition 2.1, there exist finitely many Hilbert functions $h'_i : \text{Irr}(G_x) \rightarrow \mathbb{N}$, $i = 1, 2, \dots, r$ with $h'_i(0) = 1$ and etale morphisms*

$$\text{Hilb}_{h'_i}^{G_x}(V) \rightarrow \text{Hilb}_h^G(X), \quad i = 1, 2, \dots, r$$

such that their images cover the open set $\text{Hilb}_h^G(U)$.

Moreover, if X is smooth at x , then there are etale morphisms

$$\mathrm{Hilb}_{h'_i}^{G_x}(V) \rightarrow \mathrm{Hilb}_{h_i}^{G_x}(T_x V), \quad i = 1, 2, \dots, r.$$

Under the condition $h(0) = 1$, the target of the Hilbert-Chow morphism $\mathrm{Hilb}_h^G(X) \rightarrow \mathrm{Hilb}_{h(0)}(X//G)$ is isomorphic to $X//G$. Thus under this condition, $\mathrm{Hilb}_h^G(X)$ can be regarded as a refined quotient.

The local structure of invariant Hilbert schemes at a point which represents a closed orbit is reduced to that of finite support Hilbert functions.

Theorem 2.4. *Let X be an affine G -variety. Let $x \in X(k)$ be a point such that Gx is a closed orbit and V an etale slice at x . Let $Z \subset X$ be a G -stable closed subscheme such that the underlying topological space is Gx . Let h be the Hilbert function of Z . Suppose $h(0) = 1$. Then there is a support finite Hilbert function $h' : \mathrm{Irr}(G_x) \rightarrow \mathbb{N}$ with $h'(0) = 1$ and an etale morphism*

$$\mathrm{Hilb}_{h'}^{G_x}(V) \rightarrow \mathrm{Hilb}_h^G(X)$$

such that the image contains $Z \in \mathrm{Hilb}_h^G(X)(k)$.

Moreover, if X is smooth at x , then there is an etale morphism

$$\mathrm{Hilb}_{h'}^{G_x}(V) \rightarrow \mathrm{Hilb}_{h'}^{G_x}(T_x V).$$

By Theorem 2.4, we conclude some smoothness of invariant Hilbert schemes at closed orbits.

Corollary 2.5. *Let X be an affine G -variety. Let $x \in X(k)$ be a smooth point such that Gx is a closed orbit. Let $Z \subset X$ be a G -stable closed subscheme such that the underlying topological space is Gx . Let h be the Hilbert function of Z . Suppose $h(0) = 1$.*

- (1) *If $\dim_x X - \dim Z \leq 2$, then $\mathrm{Hilb}_h^G(X)$ is smooth at Z .*
- (2) *If Z is Gorenstein and $\dim_x X - \dim Z = 3$, then $\mathrm{Hilb}_h^G(X)$ is smooth at Z .*
- (3) *If Z is reduced, then $\mathrm{Hilb}_h^G(X)$ is smooth at Z .*

Here $\dim_x X = \inf\{\dim W \mid x \in W, W \subset X \text{ open}\}$ is the dimension of X at x .

As an obvious corollary, we have the following.

Corollary 2.6. *Let X be a smooth affine G -variety and assume the action is closed i.e. for all $x \in X(k)$, its orbit Gx is closed. Let $h : \mathrm{Irr}(G) \rightarrow \mathbb{N}$ be a Hilbert function with $h(0) = 1$.*

- (1) *If $\dim X/G \leq 2$, then $\mathrm{Hilb}_h^G(X)$ is smooth.*
- (2) *If $\dim X/G \leq 3$, then $\mathrm{Hilb}_h^G(X)$ is smooth at every point which represents a Gorenstein subscheme.*
- (3) *$\mathrm{Hilb}_h^G(X)$ is smooth at every point which represents a reduced subscheme.*

Proof. Let $\pi : X \rightarrow X/G$ be the quotient morphism. Let Z be a point of $\mathrm{Hilb}_h^G(X)(k)$. Since $h(0) = 1$, $\pi(Z)$ is one point. Therefore if we take a point $x \in Z(k)$, then the underlying space of Z is Gx . Note that $\dim_x X - \dim Z \leq \dim X/G$. Now the statements follow from Corollary 2.5. \square

3. PROOFS

To prove Theorem 2.3 and Theorem 2.4, we need some lemmas.

First, we investigate that how to change invariant Hilbert schemes under the morphism of Proposition 2.1 (3).

Definition 3.1. Let X, Y be affine G -schemes of finite type over k .

- (1) An open subset $U \subset X$ is called saturated if and only if there is an open subset $V \subset X//G$ such that $\pi^{-1}(V) = U$. Here $\pi : X \rightarrow X//G$ is the categorical quotient.
- (2) A morphism $\varphi : X \rightarrow Y$ is called strongly etale if and only if

- (i) The following commutative diagram is cartesian.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ X//G & \xrightarrow{\varphi//G} & Y//G \end{array}$$

- (ii) The induced morphism $\varphi//G$ is etale.

Saturated open subsets and strongly etale morphisms are compatible with invariant Hilbert functors.

Lemma 3.2. Let X be an affine G -scheme of finite type over k , $U \subset X$ be a saturated open affine subset. For every $h : \text{Irr}(G) \rightarrow \mathbb{N}$, there is a canonical open immersion

$$\text{Hilb}_h^G(U) \rightarrow \text{Hilb}_h^G(X).$$

Proof. We consider the Hilbert schemes as functors from the category of noetherian k -schemes to the category of sets. We have to construct a natural transformation and prove it is an open immersion.

Let S be a noetherian k -scheme and $W \subset U \times S$ be a G -stable closed subscheme with Hilbert function h . Then we have the following commutative diagram.

$$\begin{array}{ccccc} & & W & & \\ & \swarrow & \downarrow i & & \\ W' & \xrightarrow{i'} & U \times S & \xrightarrow{j} & X \times S \\ \downarrow & & \downarrow & & \downarrow \\ W//G & \xrightarrow{i//G} & U//G \times S & \xrightarrow{j//G} & X//G \times S \end{array}$$

Here i (resp. j) is the inclusion, $i//G$ (resp. $j//G$) is the induced closed immersion (resp. open immersion), i' is the base change of $i//G$. Since U is saturated, j is the base change of $j//G$. Since W is multiplicity finite, $W//G$ is finite over S and $j//G \circ i//G$ is finite. Therefore $j \circ i'$ is finite and $j \circ i$ is a closed immersion. So we have an injective natural transformation $\text{Hilb}_h^G(U) \rightarrow \text{Hilb}_h^G(X)$ by sending $i : W \rightarrow U \times S$ to $j \circ i : W \rightarrow X \times S$.

Now we show that the natural transformation is an open immersion. Let $W \subset X \times S$ be a closed G -subscheme with Hilbert function h . Let $Z = W \cap (X \setminus U \times S)$, where $X \setminus U$ with its reduced structure. Then Z is a multiplicity finite closed G -subscheme of $X \times S$. Let $p : Z \rightarrow S$ be the structure morphism. Then for any morphism $f : S' \rightarrow S$,

$$f^*W \in \text{Hilb}_h^G(U)(S') \iff f^*Z = \emptyset \iff f \text{ factors } S \setminus p(Z).$$

But since $p : Z \rightarrow Z//G \rightarrow S$ and $Z//G$ is finite over S , $p(Z)$ is closed in S , hence $\text{Hilb}_h^G(U)$ is an open subfunctor of $\text{Hilb}_h^G(X)$. \square

Lemma 3.3. *Let X, Y be affine G -schemes of finite type over k , $\varphi : X \rightarrow Y$ be a strongly etale morphism. For $h : \text{Irr}(G) \rightarrow \mathbb{N}$ with $h(0) = 1$, φ induces an etale morphism*

$$\Phi : \text{Hilb}_h^G(X) \rightarrow \text{Hilb}_h^G(Y).$$

If φ is surjective, then Φ is surjective.

Proof. Step 1. First we construct a natural transformation from $\text{Hilb}_h^G(X)$ to $\text{Hilb}_h^G(Y)$. Let S be a noetherian k -scheme, and $W \subset X \times S$ be a G -stable closed subscheme with Hilbert function h . Then we have the following commutative diagram.

$$\begin{array}{ccccc} & & W & & \\ & \swarrow & \downarrow i & & \\ W' & \xrightarrow{i'} & X \times S & \xrightarrow{\varphi} & Y \times S \\ \downarrow & & \downarrow & & \downarrow \\ W//G & \xrightarrow{i//G} & X//G \times S & \xrightarrow{\varphi//G} & Y//G \times S \end{array}$$

Here i is the inclusion and i' is the base change of $i//G$. Since φ is strongly etale, φ is the base change of $\varphi//G$. By the assumption of $h(0) = 1$, $W//G = S$ and $\varphi//G \circ i//G$ is a closed immersion. So $\varphi \circ i'$ is also a closed immersion and hence $\varphi \circ i$ is a closed immersion. Therefore we get a natural transformation

$$\Phi : \text{Hilb}_h^G(X)(S) \rightarrow \text{Hilb}_h^G(Y)(S) ; (i : W \rightarrow X \times S) \mapsto (\varphi \circ i : W \rightarrow Y \times S).$$

Step 2. Next we verify that the natural transformation Φ constructed in Step 1 is an etale morphism. By the infinitesimal lifting property, we have to check that for an arbitrary noetherian affine scheme $\text{Spec} A$ over $\text{Hilb}_h^G(Y)$ and an ideal $I \subset A$ with $I^2 = 0$,

$$\text{Hom}_{\text{Hilb}_h^G(Y)}(\text{Spec} A, \text{Hilb}_h^G(X)) \rightarrow \text{Hom}_{\text{Hilb}_h^G(Y)}(\text{Spec} A/I, \text{Hilb}_h^G(X))$$

is bijective. This is equivalent to that for an arbitrary noetherian ring A over k , an ideal $I \subset A$ with $I^2 = 0$, and $Z \in \text{Hilb}_h^G(Y)(A)$,

$$\Phi(A)^{-1}(Z) \rightarrow \Phi(A/I)^{-1}(\tilde{Z})$$

is bijective. Here \tilde{Z} is the image of Z by $\text{Hilb}_h^G(Y)(A) \rightarrow \text{Hilb}_h^G(Y)(A/I)$.

$$\begin{array}{ccc} \text{Hilb}_h^G(X)(A) & \longrightarrow & \text{Hilb}_h^G(X)(A/I) \\ \Phi(A) \downarrow & & \downarrow \Phi(A/I) \\ \text{Hilb}_h^G(Y)(A) & \longrightarrow & \text{Hilb}_h^G(Y)(A/I) \end{array}$$

Consider the following commutative diagram where every square is cartesian.

$$\begin{array}{ccc}
 \tilde{T} & \xrightarrow{\beta} & \tilde{Z} \\
 \downarrow & \searrow & \swarrow \\
 & X \times \operatorname{Spec} A/I \longrightarrow Y \times \operatorname{Spec} A/I & \\
 & \downarrow \qquad \qquad \downarrow & \\
 & X \times \operatorname{Spec} A \longrightarrow Y \times \operatorname{Spec} A & \\
 \uparrow & \swarrow & \nwarrow \\
 T & \xrightarrow{\alpha} & Z
 \end{array}$$

Then

$$\begin{aligned}
 \Phi(A)^{-1}(Z) &\simeq \{\text{sections of } \alpha\} \simeq \{D \subset T \text{ open} \mid \alpha|_D \text{ is an isom}\} \\
 \Phi(A/I)^{-1}(\tilde{Z}) &\simeq \{\text{sections of } \beta\} \simeq \{\tilde{D} \subset \tilde{T} \text{ open} \mid \beta|_{\tilde{D}} \text{ is an isom}\}.
 \end{aligned}$$

Note that Z is an affine scheme and the ideal J of \tilde{Z} in Z satisfies $J^2 = 0$. Since α is etale, these two sets are canonically bijective.

Step3. Suppose φ is surjective. It is enough to show that $\operatorname{Hilb}_h^G(X)(k) \rightarrow \operatorname{Hilb}_h^G(Y)(k)$ is surjective. Let $Z \subset Y$ be a G -stable closed subscheme with Hilbert function h . Then $Z//G$ is one point and denote it $z : Z//G = \operatorname{Spec} k \rightarrow Y//G$. Consider the following commutative diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{\quad} & & & Z \\
 \downarrow & \searrow & X & \xrightarrow{\varphi} & Y & \swarrow \\
 & & \downarrow & & \downarrow & \\
 & & X//G & \xrightarrow{\varphi//G} & Y//G & \\
 \uparrow & \swarrow & & & \nwarrow & \\
 \varphi//G^{-1}(z) & \xrightarrow{\quad} & & & \operatorname{Spec} k
 \end{array}$$

where every square is cartesian. Since $\varphi//G$ is etale $\varphi//G^{-1}(z)$ is a disjoint union of $\operatorname{Spec} k$. Therefore $W \rightarrow Z$ has a section and Z is in the image of $\operatorname{Hilb}_h^G(X)(k) \rightarrow \operatorname{Hilb}_h^G(Y)(k)$. \square

Second we investigate the relation of invariant Hilbert schemes of G and G_x .

Definition 3.4. Let $H \subset G$ be a reductive subgroup, and X be an affine H -scheme of finite type over k . Then H acts on $G \times X$ by $h(g, x) = (gh^{-1}, hx)$. Then we denote the quotient $(G \times X)/H$ by $G \times^H X$ and call it the induced scheme. $G \times^H X$ is naturally a G -scheme by $g(g', x) = (gg', x)$.

Lemma 3.5. Let $H \subset G$ be a reductive subgroup, and X be an affine H -scheme of finite type. Then for every $h : \operatorname{Irr}(G) \rightarrow \mathbb{N}$, there are finitely many Hilbert functions

$h'_i : \text{Irr}(H) \rightarrow \mathbb{N}$ with $h'_i(0) = 1$ such that

$$\text{Hilb}_h^G(G \times^H X) = \coprod_i \text{Hilb}_{h'_i}^H(X)$$

Proof. This is an immediate consequence of the proof of [Br11, Lemma 3.2]. \square

Now we can easily prove Theorem 2.3.

(*proof of Theorem 2.3*). By Proposition 2.1, there is a strongly etale surjective morphism followed by saturated open immersion

$$G \times^{G_x} V \twoheadrightarrow U \hookrightarrow X.$$

Using Lemma 3.2 and Lemma 3.3 to these morphisms, we get

$$\text{Hilb}_h^G(G \times^{G_x} V) \twoheadrightarrow \text{Hilb}_h^G(U) \hookrightarrow \text{Hilb}_h^G(X)$$

where the first morphism is etale surjective and the second is an open immersion. Then the first assertion follows from Lemma 3.5. If we apply Lemma 3.3 to the strongly etale morphism $V \rightarrow T_x V$ of Proposition 2.2, then the second assertion follows. \square

The proof of Theorem 2.4 is slightly complicated because we have to chase elements of invariant Hilbert schemes.

(*proof of Theorem 2.4*). We have the following commutative diagram.

$$\begin{array}{ccc} & & Z \\ & \swarrow j & \downarrow i \\ G \times^{G_x} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ (G \times^{G_x} V) // G & \longrightarrow & X // G \end{array}$$

Here i is the inclusion and j is the closed immersion such that $j(x) = \overline{(e, x)}$ where $\overline{(e, x)}$ is the image of $(e, x) \in G \times V$ in $G \times^{G_x} V$. Next we consider the following diagram

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ Z & \xrightarrow{j} & G \times^{G_x} V & \xrightarrow{p} & G/G_x \\ \uparrow & & \uparrow & & \uparrow \\ W & \longrightarrow & V & \longrightarrow & \bar{e} \end{array}$$

where p is the morphism induced by the first projection $G \times V \rightarrow G$ and the two squares are cartesian. Note that f gives a homeomorphism of underlying spaces and especially is a quasi-finite morphism. Therefore W is finite over k and its Hilbert function h' is support finite. By construction we have

$$\text{Hilb}_{h'}^{G_x}(V) \longrightarrow \text{Hilb}_h^G(G \times^{G_x} V) \longrightarrow \text{Hilb}_h^G(X)$$

$$W \longmapsto (j : Z \rightarrow G \times^{G_x} V) \longmapsto (i : Z \rightarrow X)$$

and the first part of the theorem follows.

The proof of the second assertion is similar to that of Theorem 2.3. \square

Remark 3.6. We use the notation of the proof. If we put $n = \sum_{M \in \text{Irr}(G_x)} h'(M)$ (which is well defined because h' is support finite), we have the following commutative diagram.

$$\begin{array}{ccc} \text{Hilb}_{h'}^{G_x}(V) & \xrightarrow{\quad\quad\quad} & \text{Hilb}_n(V) \\ & \searrow & \nearrow \\ & \text{Hilb}_n(V)^{G_x} & \end{array}$$

If X is smooth at x , then we have another commutative diagram.

$$\begin{array}{ccccc} & & \text{Hilb}_h^G(X) & & \\ & \nearrow \text{etale} & & \searrow \alpha & \\ \text{Hilb}_{h'}^{G_x}(V) & \xrightarrow{\quad\quad\quad} & \text{Hilb}_{h'}^{G_x}(T_x V) & \xrightarrow{\quad\quad\quad} & \text{Hilb}_n(T_x V) \\ & \searrow \gamma \text{ etale} & & \nearrow \beta & \\ & & \text{Hilb}_n(T_x V)^{G_x} & & \end{array}$$

Here α maps W to length n closed subscheme of $T_x V$ supported at 0 and β is an open immersion. Therefore γ is an etale morphism and $\gamma(W)$ represents G_x -stable closed subscheme of $T_x V$ supported at 0.

To prove Corollary 2.5, we need some results on smoothness.

Lemma 3.7. *Let H be a reductive group over k , Y be a separated H -scheme of finite type over k . Let $y \in Y^H(k)$ and suppose Y is smooth at y . Then Y^H is also smooth at y .*

Proof. This is a well known fact (see for example [Fo73]), but for the reader's benefit we give a proof.

Let $\mathfrak{m} \subset \mathcal{O}_{Y,y}$ be the maximal ideal, $I \subset \mathcal{O}_{Y,y}$ be the ideal of Y^H in Y at y . Note that H acts $\mathcal{O}_{Y,y}$ since y is a fixed point and I is generated by $\{h \cdot f - f \mid h \in H(k), f \in \mathcal{O}_{Y,y}\}$. Since $I \cap \mathfrak{m}^2 \in I$ is a H -submodule and H is linearly reductive, there is a H -submodule $E \subset I$ such that $I = (I \cap \mathfrak{m}^2) \oplus E$. Let $J \subset I$ be the ideal generated by E .

We claim $J = I$. It is equivalent to show that H acts trivially on $\mathcal{O}_{Y,y}/J$. Let assume contrary. Then there exists $n > 0$ such that H acts non-trivially on $A := (\mathcal{O}_{Y,y}/J)/(\mathfrak{m}/J)^n$. Then we have the following commutative diagram of linear algebraic groups over k .

$$\begin{array}{ccc} H & \xrightarrow{\quad\alpha\quad} & \text{Aut}(A) \\ & \searrow \text{trivial} & \downarrow \beta \\ & & \text{GL}((\mathfrak{m}/J)/(\mathfrak{m}/J)^2) \end{array}$$

Here $\text{Aut}(A)$ is the group of k -algebra automorphisms, α is the group homomorphism induced by the action H on A and β is the natural group homomorphism.

Since H acts trivially on $\mathfrak{m}/(\mathfrak{m}^2 + J) = (\mathfrak{m}/J)/(\mathfrak{m}/J)^2$, $\beta \circ \alpha$ is trivial. Now, by the assumption, α is not trivial. But $\text{Ker} \beta$ is unipotent so that α induces non-trivial group homomorphism from a linearly reductive group to a unipotent group. This is a contradiction.

Now, take a k -basis of E . By the above claim, it generates I . It gives a basis of $I/(I \cap \mathfrak{m}^2)$ and therefore it gives a part of a basis of $\mathfrak{m}/\mathfrak{m}^2$. Hence it is a part of a regular parameter system of $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{Y,y}/I = \mathcal{O}_{Y^H,y}$ is regular. \square

On the smoothness of punctual Hilbert schemes, the following facts are known.

Proposition 3.8. *Let $\text{Hilb}_n(\mathbb{A}^d)$ be the punctual Hilbert scheme of n points.*

- (1) [Fo68] *If $d \leq 2$, then $\text{Hilb}_n(\mathbb{A}^d)$ is smooth.*
- (2) [CN09] *If $d \leq 3$ and $Z \in \text{Hilb}_n(\mathbb{A}^d)(k)$ is Gorenstein as a scheme, then $\text{Hilb}_n(\mathbb{A}^d)$ is smooth at Z .*

(proof of Corollary 2.5). Take a smooth slice V at x . By Proposition 2.2, $\dim V = \dim_x X - \dim Gx$. We use the notation of the proof of Theorem 2.4 and Remark 3.6. By Theorem 2.4 and its proof, $\text{Hilb}_h^G(X)$ is smooth at Z if and only if $\text{Hilb}_{h'}^{G_x}(V)$ is smooth at W .

(1) By Proposition 3.8 (1), $\text{Hilb}_n(T_x V)$ is smooth. By Lemma 3.7, $\text{Hilb}_n(T_x V)^{G_x}$ is smooth. Since γ is etale, $\text{Hilb}_{h'}^{G_x}(V)$ is also smooth.

(2) In the proof of Theorem 2.4, we have the following commutative diagram.

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & \curvearrowright & & \curvearrowright & \\
 G/G_x & \xrightarrow{\quad} & Z & \xrightarrow{\quad f \quad} & G/G_x \\
 & & \uparrow & & \uparrow \\
 & & W = f^{-1}(\bar{e}) & \longrightarrow & \bar{e}
 \end{array}$$

Here the morphism $G/G_x \rightarrow Z$ is the reduction. Therefore every local ring of W is the quotient of a local ring of Z by a regular sequence. Hence W is Gorenstein if (and only if) Z is Gorenstein. By Proposition 3.8 (2), $\text{Hilb}_{h'}^{G_x}(T_x V)$ is smooth at W and $\text{Hilb}_{h'}^{G_x}(T_x V)^{G_x}$ is also smooth at W . Now the statement follows from Remark 3.6.

(3) If Z is reduced, then $W = \text{Spec } k$ and h' is trivial. Therefore $\text{Hilb}_{h'}^{G_x}(V) = V^{G_x}$ and by Lemma 3.7, this is smooth. \square

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